Nonlinear Stiffness, Lyapunov Exponents, and Attractor Dimension

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Abstract

I propose that stiffness may be defined and quantified for nonlinear systems using Lyapunov exponents, and demonstrate the relationship that exists between stiffness and the fractal dimension of a strange attractor: that stiff chaos is thin chaos.

What constitutes a stiff dynamical system? Stiffness is closely related to numerical methods [1]: the signature of stiffness in a problem is that, upon integration with a general numerical scheme — a method not specially designed for stiff problems — the routine takes extremely small integration steps [2], which makes the process computationally expensive. One view is that stiffness is inextricably linked with the numerical integration scheme used, so that there would be no such thing as an intrinsically stiff dynamical system, and the best we could hope for is an operational definition such as that above [3]. Moreover, it has been proposed that chaotic problems cannot be stiff [4]. I argue below that this is not the case, and provide a definition and a quantitative measure of stiffness for nonlinear dynamical systems. I demonstrate how stiffness affects the geometry of the strange attractor of a chaotic system: that stiff chaos is thinner — has smaller fractional part of the fractal dimension — than nonstiff chaos.

When integrating a stiff problem with a variable-step explicit numerical integration scheme, the initial step length chosen causes the method to be at or near numerical instability, which leads to a large local truncation error estimate. This causes the numerical routine to reduce the step length substantially, until the principal local truncation error is brought back within its prescribed bound. The routine then integrates the problem successfully, but uses a far greater number of steps than seems reasonable, given the smoothness of the solution. Because of this, computation time and round-off error are a problem when using conventional numerical integration techniques on stiff problems, and special methods have been developed for them.

Traditionally in numerical analysis, a linear stiff system of size n is defined

by [5]

$$\mathrm{Re}(\lambda_i) < 0, \qquad 1 \leq i \leq n, \quad (1)$$

where λ_i are the eigenvalues of the Jacobian of the system, with

$$\max_{1 \leq i \leq n} |\mathrm{Re}(\lambda_i)| \gg \min_{1 \leq i \leq n} |\mathrm{Re}(\lambda_i)|. \quad (2)$$

The *stiffness ratio* R provides a quantitative measure of stiffness:

$$R = \frac{\max\limits_{1 \leq i \leq n} |\mathrm{Re}(\lambda_i)|}{\min\limits_{1 \leq i \leq n} |\mathrm{Re}(\lambda_i)|}. \quad (3)$$

By this definition, a stiff problem has a stable fixed point with eigenvalues of greatly differing magnitudes; large negative eigenvalues correspond to fast-decaying transients $e^{\lambda t}$ in the solution.

As an example of a linear stiff problem, consider the equation

$$y'' + 1001y' + 1000y = 0. \quad (4)$$

We can write this as the vector equation $\mathbf{y}' = A\mathbf{y}$ where

$$A = \begin{pmatrix} 0 & 1 \\ -1000 & -1001 \end{pmatrix}, \quad (5)$$

and the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -1000$. This has solution

$$y = Ae^{-t} + Be^{-1000t}, \quad (6)$$

so when integrating the problem numerically with a general variable-step method we would expect to be able to use a large integration step length after

the e^{-1000t} transient term decays, but in fact the presence of the large negative eigenvalue λ_2 prevents this. With appropriate initial conditions, one can even set $B = 0$ and remove the e^{-1000t} term from the solution entirely; one nevertheless has to use a very small step length throughout the calculation, as step length is still dictated by the size of λ_2.

The definition of linear stiffness of Eqs. (2) and (3) is not relevant for nonlinear systems. The stiffness ratio Eq. (3) is often not a good measure of stiffness even for linear systems, since if the minimum eigenvalue is zero, the problem has infinite stiffness ratio, but may not really be stiff at all if the other eigenvalues are of moderate size. The inadequacy of the stiffness ratio is clearly recognized by numerical analysts, who have moved away from trying to pin down the definition of stiffness and have generally adopted the pragmatic approach I alluded to earlier: "Stiff equations are problems for which explicit methods don't work" [3].

Let us look at a nonlinear stiff problem. In Fig. 1, I show the results of integrating the van der Pol equation

$$y'' - \mu(1 - y^2)y' + y = 0 \quad (7)$$

using a standard variable-step fourth-order Runge–Kutta code [6], for the cases $\mu = 1$ and $\mu = 100$. There is the same bound on the principal local truncation error estimate in both cases. We can see from the far greater number of steps needed for $\mu = 100$, that at large μ the van der Pol equation becomes very stiff. The steps are so
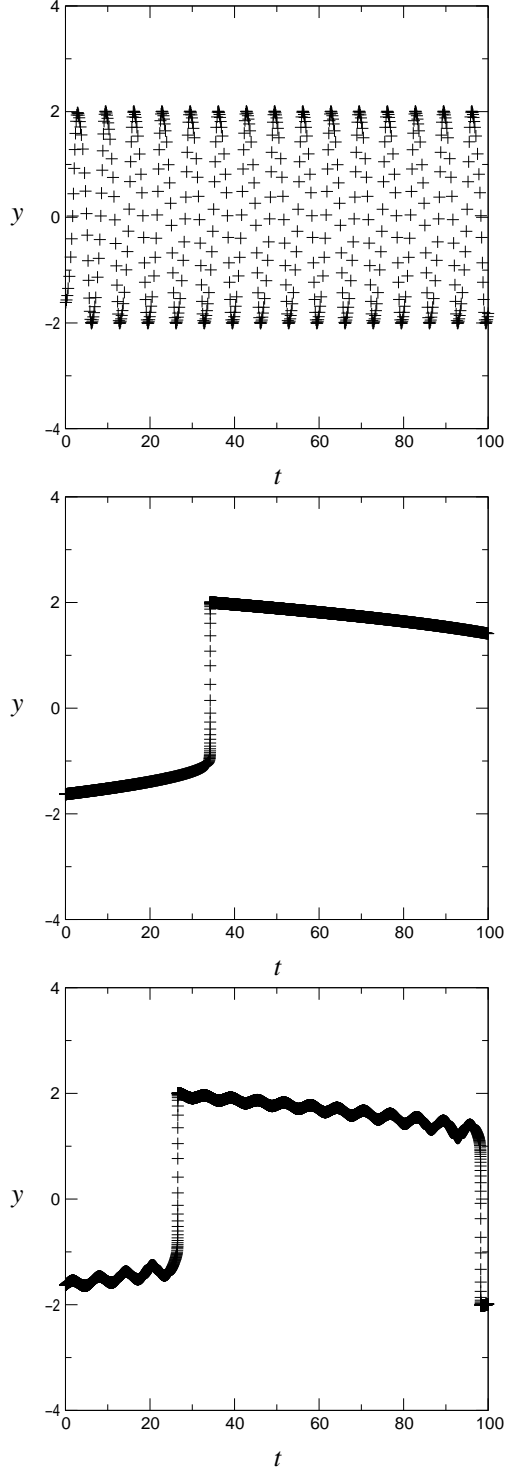
Fig. 1. Numerical integrations using a variable-step fourth-order Runge–Kutta method. The principal local truncation error is $\varepsilon = 10^{-4}$ in all cases. The van der Pol equation with (a) $\mu = 1$ and (b) $\mu = 100$; (c) forced van der Pol equation with $\mu = 100$, $A = 10$, and $\omega = 1$.

small at $\mu = 100$ that the individual crosses representing each step merge to form a continuous broad line on the graph. At these large μ values, the equation describes a relaxation oscillator [7]. These have fast and slow states in their cycle which characterizes the jerky motion displayed in Fig. 1(b). And if we introduce forcing into the van der Pol equation

$$y'' - \mu(1 - y^2)y' + y = A\cos\omega t \quad (8)$$

we obtain a chaotic system [8–11] which, like its unforced counterpart, is stiff, as demonstrated in Fig. 1(c)), where the steps are so small that they have merged together in the image.

Throughout the foregoing, I have illustrated stiff systems with reference to their effect on the numerical method used for integrating them. Is it possible then to define nonlinear stiffness independently of the numerical method used? Lambert [5] puts forward the definition: *A system is said to be stiff in a given interval x if in that interval the neighbouring solution curves approach the solution curve at a rate which is very large in comparison with the rate at which the solution varies in that interval.* This definition does not make reference to the numerical method used and is instead concerned with the curvature and local Lyapunov exponents of the solution. The curvature of the solution curve

$$\kappa = \frac{y''}{(1 + y'^2)^{3/2}} \quad (9)$$

at a point quantifies the wiggliness of the trajectory at that point, while local

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Lyapunov exponents are defined as

$$\gamma_i(\tau, t) = \lim_{\sigma_i(\tau) \to 0} \frac{1}{\tau} \ln \frac{\sigma_i(t+\tau)}{\sigma_i(t)}, \quad (10)$$

where $\sigma_i(t)$ are the principal axes of an ellipsoidal ball evolving in time in phase space. The local Lyapunov exponent $\gamma_i(\tau, t)$ is a generalization of the Lyapunov exponent λ_i: the Lyapunov exponent is the limit of τ going to infinity in the local Lyapunov exponent

$$\lambda_i = \lim_{\tau \to \infty} \gamma_i(\tau, t). \quad (11)$$

Lyapunov exponents show the rate of convergence or divergence of neighbouring trajectories, and an n-dimensional system has n Lyapunov exponents corresponding to n expanding or contracting directions in phase space. Since Lyapunov exponents are defined in the infinite time limit, they cannot reflect differing rates of convergence or divergence in different parts of a trajectory. Whereas the Lyapunov exponent is the same for almost all starting points on a trajectory, the local Lyapunov exponent can vary depending on the starting point and the length of trajectory examined. Fast convergence to a neighbouring trajectory, which implies having large negative local Lyapunov exponents, indicates stiffness. A system that is equally stiff at all points along a trajectory has a constant convergence rate, and in this case, the local Lyapunov exponent will be the same as the Lyapunov exponent. Often, though, a system can show intervals of stiff and nonstiff behaviour. Relaxation oscillators like the van der Pol oscillator are examples of this, having a stiff slow manifold, and a nonstiff fast manifold. With each oscillation we have two intervals of stiff behaviour interspersed with two intervals of nonstiff behaviour.

As it stands, Lambert's definition of stiffness will not do for us, since it assumes that the system is not chaotic: the principal local Lyapunov exponent is large and negative to obtain fast convergence of neighbouring trajectories. Instead, we need to adapt it to allow for chaotic behaviour along with stiffness, by looking at the largest negative local Lyapunov exponent, rather than the principal local Lyapunov exponent. Our definition of nonlinear stiffness is then: *A system is stiff in a given interval if in that interval the most negative local Lyapunov exponent is large, while the curvature of the trajectory is small.* A quantitative measure of nonlinear stiffness at any point can then be obtained from the ratio of the most negative local Lyapunov exponent and the curvature of the trajectory:

$$R_{nl} = \frac{\left| \min_{1 \le i \le n} \gamma_i(\tau, t) \right|}{\kappa(t)}. \quad (12)$$

If desired this could be averaged over the trajectory to give a measure of mean stiffness.

We now have a definition of nonlinear stiffness; what does it imply for chaotic systems? If a Lyapunov exponent of a system is large and negative, then the local Lyapunov exponent must be large and negative at least over some of the trajectory, so, given suitable bounds on the curvature of the trajectory, a large negative Lyapunov exponent is a sufficient condition for stiffness. On the

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other hand, at least one positive Lyapunov exponent is necessary for chaos. Thus for stiff chaos we should have a large spread of Lyapunov exponents, with at least one positive, and one large and negative.

Lyapunov exponents are certainly related to the fractal dimension of an attractor. The Kaplan–Yorke conjecture [12] holds that the fractal dimension is

$$D_{KY} = \frac{1}{|\lambda_{j+1}|} \sum_{k=1}^{j} \lambda_k + j, \qquad (13)$$

where j is the expansion dimension of the system: the largest integer such that

$$\sum_{k=1}^{j} \lambda_k \geq 0. \qquad (14)$$

The Kaplan–Yorke estimate is often good — i.e., very close to the measured (capacity) dimension. In general the fractal dimension D lies between the expansion dimension and the Kaplan–Yorke estimate [13]:

$$j \leq D \leq D_{KY}. \qquad (15)$$

For example, in a three-dimensional chaotic system, $\lambda_1 > 0$, $\lambda_2 = 0$, and $\lambda_3 < 0$, so

$$D_{KY} = 2 + \frac{\lambda_1}{|\lambda_3|}. \qquad (16)$$

From this, if $|\lambda_3|$ gets larger, with increasing stiffness, then the fractal dimension of the strange attractor moves closer to two.

The size of the fractional part of the fractal dimension of a strange attractor may be termed the thickness or thinness of the chaos. The folded structure of the foliations of the strange attractor — the Abraham and Shaw bagel [14] — is tightly wound in the case of thin chaos with small fractional part. On the other hand, if the fractal folding of the surface of the strange attractor is macroscopic, the chaos is thick chaos having a large fractional part to its fractal dimension.

We should expect then that stiff chaos will be thin, and this indeed proves to be the case. In Fig. 1 I illustrated the stiffness of the forced van der Pol equation. This equation is also well known for the difficulty of capturing pictorially its chaotic nature at large μ [15]. On the other hand, with the Shaw variant of the forcing [16], which may be written as a three-dimensional autonomous system

$$
\begin{aligned}
\dot{x} &= -y + \mu(1 - y^2)x \\
\dot{y} &= x + A\cos\omega z \\
\dot{z} &= 1
\end{aligned}
\qquad (17)
$$

whose divergence — the sum of the Lyapunov exponents — is $\mu(1 - y^2)$, a strange attractor is easily found at $A = 0.932$, $\mu = 1.18$, and $\omega = 1.86$. Now, with the benefit of the above analysis, we see how fractal dimension, dissipation, and stiffness are all linked together: whereas the normal forced van der Pol equation studied at large μ (high dissipation) is stiff, making the chaos thin, the attractor in the Shaw variant at small μ (low dissipation) displays thick chaos, and, following these arguments, will not be stiff; this is indeed the case (Fig. 2).
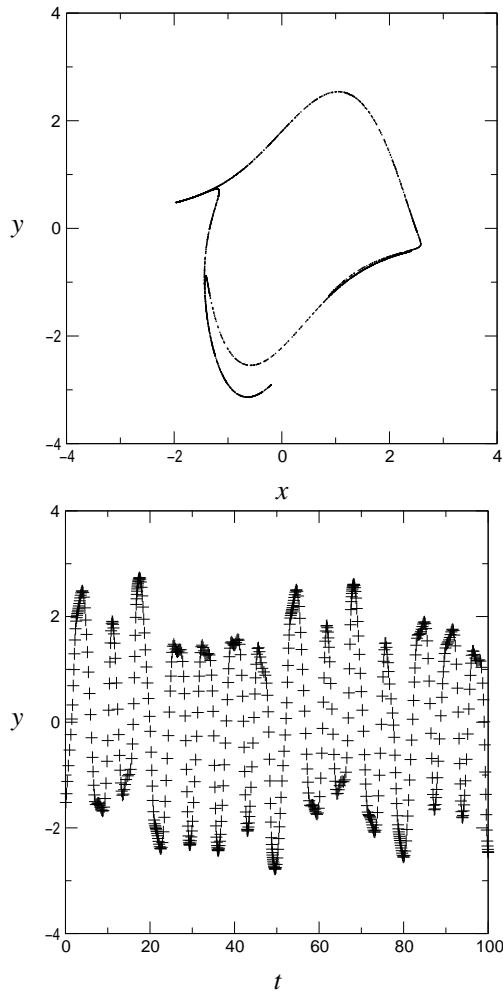
Fig. 2. The Shaw version of the forced van der Pol equation, parameters $A = 0.932$, $\mu = 1.18$, and $\omega = 1.86$. (a) Poincaré map showing strange attractor. (b) Numerical integration using a variable-step fourth-order Runge–Kutta method [6], with the same principal local truncation error $\varepsilon = 10^{-4}$ as in Fig. 1.

References

[1] J. H. E. Cartwright and O. Piro. The dynamics of Runge–Kutta methods. *Int. J. Bifurcation and Chaos*, 2:427–49, 1992.

[2] R. C. Aiken, editor. *Stiff Computation*. Oxford University Press, 1985.

[3] E. Hairer and G. Wanner. *Solving Ordinary Differential Equations II: Stiff and Differential–Algebraic Problems*. Springer, 2nd edition, 1996.

[4] R. M. Corless. Defect-controlled numerical methods and shadowing for chaotic differential equations. *Physica D*, 60:323–334, 1992.

[5] J. D. Lambert. *Numerical Methods for Ordinary Differential Systems*. Wiley, 1991.

[6] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. A. Vetterling. *Numerical Recipes in C*. Cambridge University Press, 2nd edition, 1992.

[7] B. van der Pol. On relaxation-oscillations. *Phil. Mag. (7)*, 2:978–992, 1927.

[8] M. L. Cartwright and J. E. Littlewood. On nonlinear differential equations of the second order. I. The equation $\ddot{y} + k(1 - y^2)\dot{y} + y = b\lambda k \cos(\lambda t + a)$, k large. *J. Lond. Math. Soc.*, 20:180–189, 1945.

[9] E. A. Jackson. *Perspectives of Nonlinear Dynamics*, volume 1. Cambridge University Press, 1989.

[10] K. Tomita. Periodically forced nonlinear oscillators. In A. V. Holden, editor, *Chaos*. Manchester University Press, 1986.

[11] J. M. T. Thompson and H. B. Stewart. *Nonlinear Dynamics and Chaos*. Wiley, 1986.

[12] J. L. Kaplan and J. A. Yorke. *Chaotic Behavior of Multidimensional Difference Equations*, volume 730 of *Lecture Notes in Mathematics*, pages 204–227. Springer, 1979.

[13] P. Grassberger and I. Procaccia. Measuring the strangeness of strange attractors. *Physica D*, 9:189–208, 1983.

[14] R. H. Abraham and C.D. Shaw, editors. *Dynamics: The Geometry of Behavior*. Addison-Wesley, 1992.

[15] E. A. Jackson. *Perspectives of Nonlinear Dynamics*, volume 2. Cambridge University Press, 1990.

[16] R. Shaw. Strange attractors, chaotic behavior, and information flow. *Z. Naturf.*, 36a:80–112, 1981.